12.5 Lines and Planes in 3D

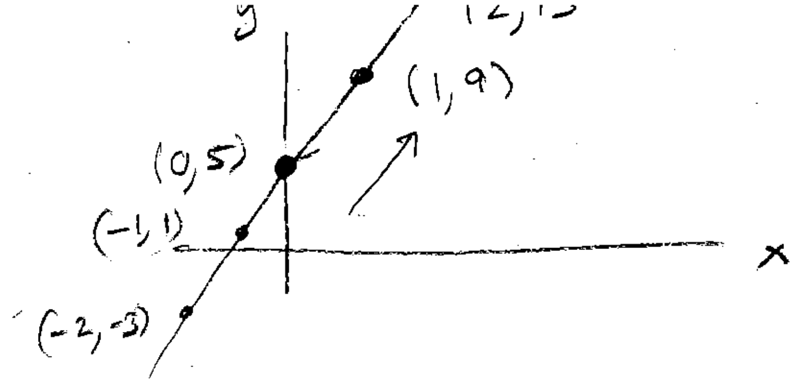
Lines: We use parametric equations for 3D lines. Here's a 2D warm-up:

Consider the line: y = 4x + 5.

(a) Find a vector parallel to the line. Call it **v**.

(b) Find a vector whose head touches the line when drawn from the origin. Call it $\mathbf{r_0}$.

(c) Observe, we can reach all other points on the line by walking along $\mathbf{r_0}$, then adding scale multiples of **v**.



$\vec{v} = \langle 1,4 \rangle$ (or $\langle 2,8 \rangle$ or $\langle 3,12 \rangle$ etc..)

$\vec{r}_0 = \langle 0,5 \rangle$ (or $\langle -1,1 \rangle$ or $\langle 1,9 \rangle$ etc..)

Let (x,y) = ANOTHER POINT ON THE LINE

THEN SOME CONSTANT MULTIPLE

$$\langle x,y \rangle = \langle 0,5 \rangle + t\langle 1,4 \rangle$$

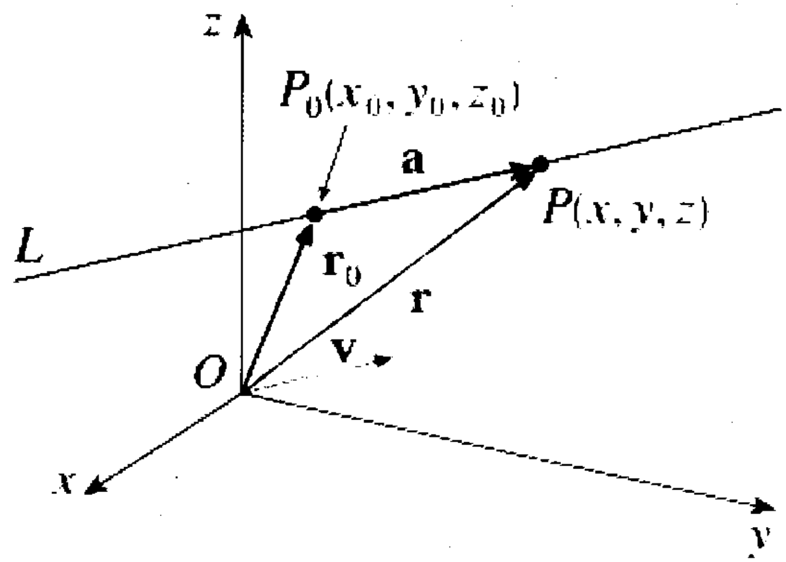
$$\Rightarrow \underbrace{x = 0 + t, \quad y = 5 + 4t}_{\text{ONE PARAMETRIZATION}}$$

Ex) $(1,9) \iff t=1$; $(2,13) \iff t=2$; $(-1,1) \iff t=-1$ — FOR THIS PARAMETRIZATION

This same idea works to describe any line in 2- or 3-dimensions.

Summary of Line Equations

Let (x,y,z) be any point on the line and $\boldsymbol{r} =< x, y, z >$ = a vector pointing to this point from the origin.



Find **a direction vector** and **a point** on the line.

1. $\boldsymbol{v} = \langle a, b, c \rangle$ *direction vector*
2. $\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ *position vector*

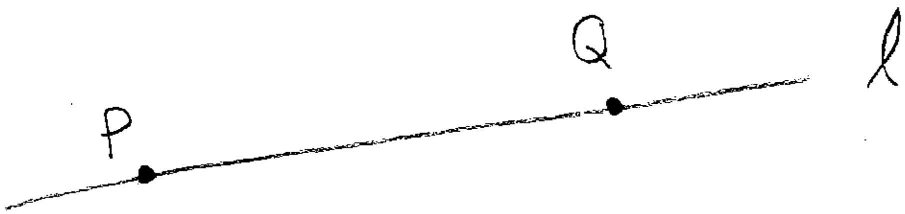
$$\boldsymbol{r} = \boldsymbol{r_0} + \mathrm{t}\,\boldsymbol{v}$$ *vector form*

$$(x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct)$$ *parametric form*

$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct.$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$ *symmetric form*

Basic Example – Given Two Points:
Find parametric equations of the line thru P(3, 0, 2) and Q(-1, 2, 7).



ASIDE

IS (11, -4, -2) ON THIS LINE?

$11 = 3 - 4t \Leftrightarrow t = -2$
$-4 = 0 + 2t \Leftrightarrow t = -2$
$-2 = 2 + 5t \Leftrightarrow t = 0$
(No!)

(11, -4, -8) IS ON THE LINE

NEED

$\boxed{1}$ A POINT: $\vec{r}_0 = \langle 3, 0, 2 \rangle$ (OR COULD USE (-1, 2, 7))

$\boxed{2}$ A DIRECTION: $\vec{v} = \overrightarrow{PQ} = \langle -1-3, 2-0, 7-2 \rangle = \langle -4, 2, 5 \rangle$ (OR COULD USE $\overrightarrow{QP}$)

$$\boxed{\langle x, y, z \rangle = \langle 3, 0, 2 \rangle + t \langle -4, 2, 5 \rangle}$$ ← VECTOR FORM

$$\boxed{\begin{aligned} x &= 3 - 4t \\ y &= 0 + 2t \\ z &= 2 + 5t \end{aligned}}$$ ← PARAMETRIC FORM

$$\underbrace{t = \frac{x-3}{-4}, \quad t = \frac{y-0}{2}, \quad t = \frac{z-2}{5}}$$

$$\boxed{\frac{x-3}{-4} = \frac{y-0}{2} = \frac{z-2}{5}}$$

Symmetric form ↑

General Line Facts

1. Two lines are **parallel** if their direction vectors are parallel.

EX) FIND THE EQUATION FOR THE LINE THRU $(1,5,-2)$ AND PARALLEL TO L1

L1: $x = 4t$, $y = 1 - t$, $z = 7 + 2t$

ANSWER: $\boxed{x = 1 + 4t,\ y = 5 - t,\ z = -2 + 2t}$

2. Two lines **intersect** if they have an (x,y,z) point in common.
Use different parameters when you combine!

Ex) L1: $x = t$, $y = 1 + 2t$, $z = 2 - 3t$ L2: $x = 3 - 4u$, $y = 2 - 3u$, $z = 1 + 2u$

(i) $t \stackrel{?}{=} 3 - 4u$
(ii) $1 + 2t \stackrel{?}{=} 2 - 3u$

$1 + 2(3 - 4u) = 2 - 3u$
$\Rightarrow 1 + 6 - 8u = 2 - 3u$
$\Rightarrow 5 = 5u$ $\boxed{u = 1}$
$\Rightarrow \boxed{t = -1}$

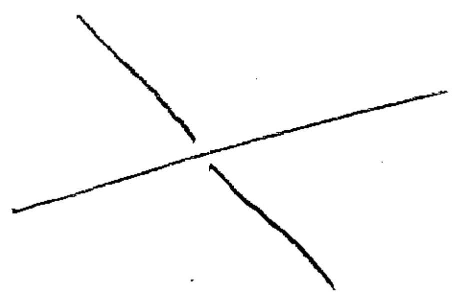
(iii) $2 + 3t \stackrel{?}{=} 1 + 2u$
$-1 \neq 3$

NO DO NOT INTERSECT

Note: The *acute angle of intersection* is the acute angle between the direction vectors.

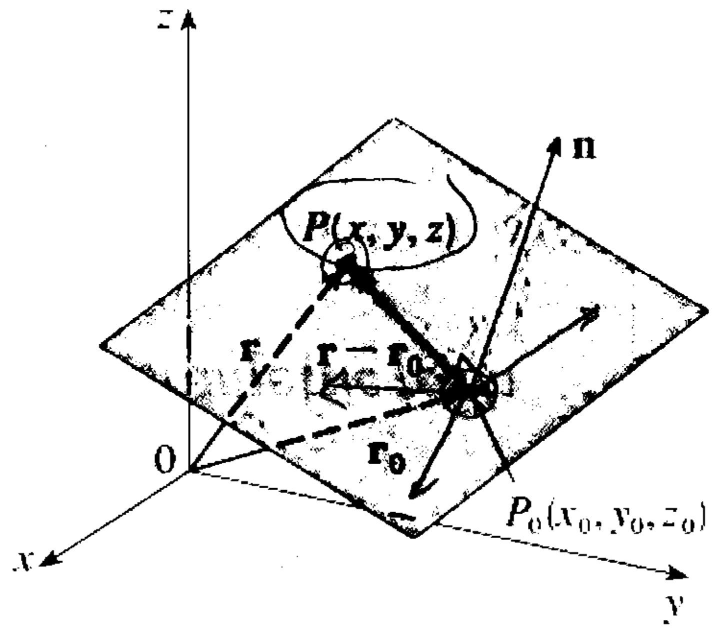
$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2|\cos\theta$

3. Two lines are **skew** if they don't intersect and aren't parallel.



Summary of Plane Equations

Let (x,y,z) be any point on the plane and $\boldsymbol{r} =< x, y, z >$ = a vector pointing to this point from the origin.



Find <u>**a normal vector**</u> and <u>**a point**</u> on the plane.

1. $\boldsymbol{n} = \langle a, b, c \rangle$ *normal vector*

2. $\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ *position vector*

Normal OTHER PT GIVEN POINT

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$$ *vector form*

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

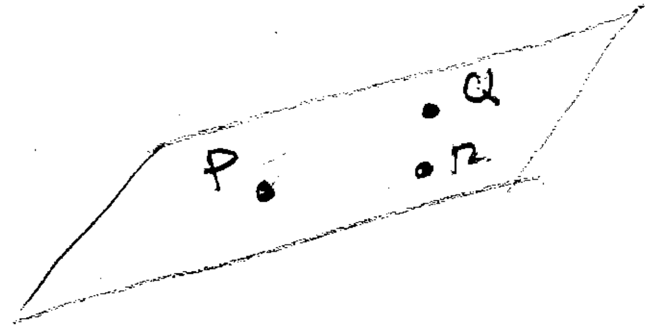
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$ *standard form*

If you expand out standard form you can write:

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$ax + by + cz = d$$, where $d = ax_0 + by_0 + cz_0$

Basic Example – Given Three Points:
Find the equation for the plane through the points P(0, 1, 0), Q(3, 1, 4), and R(-1, 0, 0)



NEED

$\boxed{1}$ A POINT: $\vec{r}_0 = \langle 0,1,0\rangle$ (OR $\langle 3,1,4\rangle$ OR $\langle -1,0,0\rangle$)

$\boxed{2}$ A NORMAL: $\overrightarrow{PQ} = \langle 3,0,4\rangle$, $\overrightarrow{PR} = \langle -1,-1,0\rangle$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 4 \\ -1 & -1 & 0 \end{vmatrix} = (0 - -4)\vec{i} - (0 - -4)\vec{j} + (-3 - 0)\vec{k}$$

$$= \langle 4, -4, -3\rangle$$

CHECK!
$12 + 0 - 12 = 0$ ✓
$-4 + 4 + 0 = 0$ ✓

$$\boxed{4(x-0) - 4(y-1) - 3(z-0) = 0}$$

$$4x - 4y + 4 - 3z = 0$$

$$4x - 4y - 3z = \underset{d}{-4}$$

General Plane Facts

1. Two planes are **parallel** if their normal vectors are parallel.

EX) GIVEN $3x + 4y + 10z = 14$

FIND THE PLANE THAT IS PARALLEL AND THRU $(7, 15, -8)$

ANSWER: $3(x-7) + 4(y-15) + 10(z+8) = 0$

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2. If two planes are not parallel, then they must intersect to form a line.

   2a. The *acute angle of intersection* is the acute angle between their normal vectors.

   $$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta$$

   2b. The planes are orthogonal if their normal vectors are orthogonal.

EX)

$\boxed{P1}$: $2x + y - z = 10$

$\boxed{P2}$: $x - y + 3z = 2$

FIND TWO POINTS OF INTERSECTION.

$\boxed{1}$ COMBINE $\Rightarrow$ $3x + 2z = 12$

$\boxed{2}$ YOU PICK POINTS! $\Rightarrow$ $x = 0 \Rightarrow z = 6$; $z = 0 \Rightarrow x = 4$

I PICKED THESE!

$\boxed{3}$ GO BACK TO GET y $\Rightarrow$ $x = 0,\ z = 6 \Rightarrow \left.\begin{array}{l} 0 + y - 6 = 10 \\ 0 - y + 18 = 2 \end{array}\right\} \Rightarrow y = 16$

$z = 0,\ x = 4 \Rightarrow \left.\begin{array}{l} 8 + y - 0 = 10 \\ 4 - y + 0 = 2 \end{array}\right\} \Rightarrow y = 2$

HERE ARE TWO POINTS ON INTERSECTION

(0, 16, 6)
(4, 2, 0)

12.5 Summary

Lines: Find a POINT and DIRECTION.

$\boldsymbol{v} = \langle a, b, c \rangle$ *direction vector*

$\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ *position vector*

$x = x_0 + at, y = y_0 + bt, z = z_0 + ct.$

Planes: Find a POINT and NORMAL

$\boldsymbol{n} = \langle a, b, c \rangle$ ***normal*** vector

$\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ *position vector*

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

To find equations for a line

Info given?	Find two points	
Done.	$\vec{v} = \overrightarrow{AB}$ (subtract components)	$\overrightarrow{r_0} = \vec{A}$

lines parallel – directions parallel.

lines intersect – make (x,y,z) all equal (different param!)

Otherwise, we say they are skew.

To find the equation for a plane

Info given?	Find three points	
Done.	Two vectors parallel to the plane: $\overrightarrow{AB}$ and $\overrightarrow{AC}$	
	$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$	$\overrightarrow{r_0} = \vec{A}$

planes parallel – normals parallel.

Otherwise, the planes intersect.